NOTES

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Modifications of Thomae's Function and Differentiability

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1. INTRODUCTION. Dirichlet constructed what is now the best known, in fact canonical, example of a nowhere continuous function:

$$D(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases}$$

In 1875, K. J. Thomae modified this example to produce a function that is continuous on the irrationals and discontinuous on the rationals (see [**1**, p. 102] for proofs):

$$T(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/n & \text{if } x = m/n \text{ where } m \text{ and } n \text{ are coprime}, \\ 1 & \text{if } x = 0. \end{cases}$$

As we shall see later, Thomae's function is not differentiable on the irrationals. In this note, we address whether there is a modification of Thomae's function which is differentiable on a subset of the irrationals.

In Section 2, we prove that Thomae's function is not differentiable on the irrationals and define modified versions of Thomae's function. In Section 3, we show that for each of our modifications there is a dense subset of irrationals on which, quite surprisingly, the function is not differentiable. Finally, in Section 4, we show that the measure of irrationality of a given number determines which modifications of Thomae's function are differentiable at that number.

2. MODIFIED THOMAE FUNCTION. The fact that T is not differentiable on the irrationals derives from the following trivial fact: for all $a \in \mathbb{R} \setminus \mathbb{Q}$ and for each $n \in \mathbb{N}$ there exists a $j_n \in \mathbb{Z}$ such that $|j_n/n - a| \leq 1/n$. By definition, $T(j_n/n) \geq 1/n$. It follows that

$$\frac{|T(j_n/n) - T(a)|}{|j_n/n - a|} = \frac{T(j_n/n)}{|j_n/n - a|} \geq 1 \text{ for all } n.$$

Since $j_n/n \to a$ as $n \to \infty$, this rational approximation of a yields that the derivative cannot be zero. However, by irrational approximation, if it exists it must be zero.

This proof relies on the fact that T sends m/n to $1/n$, making the approximation of j_n/n to a sufficiently close. If, for example, m/n is sent to $1/n^2$, the approximation of j_n/n to a is no longer close enough to ensure that the function is not differentiable at a. This led us to make the following general definition:

Let (a_i) be a sequence of reals decreasing to zero. Define the modified Thomae function with respect to (a_i) as follows:

$$T_{(a_i)}(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ a_n & \text{if } x = m/n \text{ where } m \text{ and } n \text{ are coprime}, \\ 1 & \text{if } x = 0. \end{cases}$$

Since $\lim_n a_n = 0$, $T_{(a_n)}$ is continuous on the irrationals. The faster the sequence (a_i) tends to zero, the larger the set of irrationals on which $T_{(a_i)}$ will be differentiable.

3. A DENSE SET. While attempting to prove that $T_{(1/n^2)}$ is differentiable on the irrationals, we discovered that quite the opposite is actually true. In fact, as the following proposition indicates, functions that are zero on the irrationals and positive on the rationals will always be non-differentiable on a rather large set.

Proposition 3.1. *Let f be a function on $\mathbb{R}$ that is positive on the rationals and* 0 *on the irrationals. Then there is an uncountable dense set of irrationals on which f is not differentiable.*

Proof. Let (r_i) be an enumeration of the rationals. We recursively define a convergent sequence of rationals. Let $x_1 \in \mathbb{Q}$. Find a closed interval I_1 such that for all $x \in I_1$,

$$f(x_1) \geq |x_1 - x|.$$

Having defined I_n and x_n, define x_{n+1} and I_{n+1} such that:

1. $I_{n+1} \subset I_n$;
2. length$(I_{n+1}) < 1/n$;
3. $x_{n+1} \in I_{n+1} \cap \mathbb{Q}$ and for all $x \in I_{n+1}$,

$$f(x_{n+1}) \geq |x_{n+1} - x|;$$

4. $r_i \notin I_{n+1}$ for $i = 1, \ldots, n$.

The intervals $(I_n)_{n=1}^{\infty}$ are nested nonempty intervals whose diameters converge to zero. Thus, $\bigcap_{n=1}^{\infty} I_n = \{a\}$ where $x_i \to a$ and $a \notin \mathbb{Q}$ (by (4)). If f were differentiable at a, by irrational approximation of a, the derivative would have to be zero. However since $a \in I_i$,

$$\frac{|f(x_i) - f(a)|}{|x_i - a|} = \frac{f(x_i)}{|x_i - a|} \geq 1$$

for all $i \in \mathbb{N}$. Thus, f is not differentiable at a. A look at our construction shows that the set A of all points found in this manner is dense.

To show that A is uncountable, we assume that it is not and let (b_i) be an enumeration of A. By going through the procedure outlined above and including the restriction $b_i \notin I_{n+1}$ for $i = 1, \ldots, n$, we produce an $a \notin A$ on which f is not differentiable. Therefore A must be uncountable. ■

As a corollary, no matter how quickly the sequence (a_i) converges to zero (e.g., $a_i = 1/i^{i^i}$), there is always an uncountable dense subset on which $T_{(a_i)}$ is not differentiable.

Corollary 3.2. *Let (a_i) be a sequence of reals decreasing to zero. There is an uncountable dense subset $A_{(a_i)}$ of $\mathbb{R} \setminus \mathbb{Q}$ such that $T_{(a_i)}$ is not differentiable on $A_{(a_i)}$.*

On the other hand, if you specify a countable set from the beginning, it is possible to construct a function which is positive on the rationals, 0 on the irrationals, and differentiable on that countable set.

Proposition 3.3. *Let $\{a_i : i \in \mathbb{N}\} \subset \mathbb{R} \setminus \mathbb{Q}$. Then there is a function that is positive on the rationals,* 0 *on the irrationals, and differentiable on the set $\{a_i : i \in \mathbb{N}\}$.*

Proof. For each $i \in \mathbb{N}$ define $g_i(n) = \min\{|m/n - a_i| : m \text{ and } n \text{ are coprime}\}$ and $g(n) = \min_{i \leq n} g_i(n)$. The function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ (g(n))^2 & \text{if } x = m/n \text{ where } m \text{ and } n \text{ are coprime}, \\ 1 & \text{if } x = 0 \end{cases}$$

is differentiable on $\{a_i : i \in \mathbb{N}\}$. To see this, fix $i \in \mathbb{N}$. For m and n coprime with $n \geq i$,

$$\frac{|f(m/n) - f(a_i)|}{|m/n - a_i|} = \frac{f(m/n)}{|m/n - a_i|} \leq \frac{(g(n))^2}{g_i(n)} \leq \frac{(g_i(n))^2}{g_i(n)} = g_i(n).$$

Since $g_i(n) \to 0$ as $n \to \infty$, the claim is proved. ■

4. MEASURE OF IRRATIONALITY. By Corollary 3.2 it is not possible for any of the modified Thomae functions to be differentiable on all the irrationals. Let (a_i) be a sequence of reals decreasing to zero. In this section, we study the question: On which irrationals, if any, is $T_{(a_i)}$ differentiable? To address this question, we use Diophantine approximation theory.

We begin by recalling the notion of irrationality measure. For an irrational number a define

$$M(a) = \left\{\mu > 0 : \left|a - \frac{m}{n}\right| < \frac{1}{n^\mu} \text{ has at most finitely many solutions } m \text{ and } n\right\},$$
$$\mu(a) = \inf M(a).$$

Notice that $|a - m/n| < 1/n^{\mu(a)}$ may hold for infinitely many m and n. If $M(a) = \emptyset$, define $\mu(a) = \infty$. Irrational numbers a for which $\mu(a) = \infty$ are called Liouville numbers.

Lagrange showed [**2**, Theorem 193, p. 164] that any irrational number a has infinitely many rational approximations m/n satisfying $|a - m/n| < 1/n^2$. Thus, for all irrational a, $\mu(a) \geq 2$. The fundamental theorem in Diophantine approximation of algebraic irrationals, called Roth's Theorem [**7**], states that for an algebraic irrational a, $\mu(a) = 2$. Researchers have computed numerous upper bounds on the measure of irrationality for many known transcendental (or conjecturally transcendental) numbers. For example: $\mu(e) = 2$, $\mu(\pi) < 8.0161$ [**4**], $\mu(\pi^2) < 5.441243$ [**6**], $\mu(\ln(2)) < 3.8913998$ [**3, 8**], and $\mu(\zeta(3)) < 5.513891$ [**5**].

Our first application follows from Lagrange's result.

Proposition 4.1. *Let (a_i) be a sequence of reals decreasing to zero such that $\liminf_n a_n n^2 > 0$. Then $T_{(a_i)}$ is nowhere differentiable.*

Proof. Let $a \in \mathbb{R} \setminus \mathbb{Q}$. By Lagrange's result,

$$\frac{|T_{(a_i)}(m/n) - T_{(a_i)}(a)|}{|m/n - a|} = \frac{T_{(a_i)}(m/n)}{|m/n - a|} > n^2 a_n$$

holds for infinitely many coprime m and n. Since $\liminf_n a_n n^2 > 0$, there is a sequence of rationals approaching a which ensures that $T'_{(a_i)}(a)$ is not zero. This contradicts the fact that $T'_{(a_i)}(a)$ must be zero through irrational approximation. ■

Let (a_i) be a sequence of reals decreasing to zero. Define

$$E(a_i) := \left\{\alpha \in \mathbb{R} : \alpha \geq 1 \text{ and } \lim_n a_n n^\alpha = 0\right\}, \ \alpha(a_i) := \sup E(a_i),$$

$$F(a_i) := \left\{\beta \in \mathbb{R} : \beta \geq 1 \text{ and } \liminf_n a_n n^\beta = 0\right\}, \ \beta(a_i) := \sup F(a_i).$$

Here we use the convention $\sup \emptyset = -\infty$. Notice that $E(a_i) \subseteq F(a_i)$ and thus $\alpha(a_i) \leq \beta(a_i)$. For example, $\alpha(e^{-n}) = \beta(e^{-n}) = \infty$, for $k > 1$ $\alpha(n^{-k}) = \beta(n^{-k}) = k$, and for the sequence (a_i) defined by $a_1 = a_2 = 1$ and

$$a_n = \frac{1}{2^{2^{k+1}}} \text{ for } k \text{ such that } 2^{2^{k-1}} + 1 \leq n \leq 2^{2^k},$$

$\beta(a_i) = 4$ and $\alpha(a_i) = 2$.

Proposition 4.2. *Let (a_i) be a sequence of reals decreasing to zero and $a \in \mathbb{R} \setminus \mathbb{Q}$.*

1. *If $\mu(a) < \alpha(a_i)$, then $T_{(a_i)}$ is differentiable at a.*
2. *If $\mu(a) > \beta(a_i)$, then $T_{(a_i)}$ is not differentiable at a.*

Proof. Fix a sequence (a_i) of reals decreasing to zero and $a \in \mathbb{R} \setminus \mathbb{Q}$. Suppose $\mu(a) < \alpha(a_i)$. Let $\varepsilon > 0$ such that $\mu(a) + \varepsilon < \alpha(a_i)$. For n sufficiently large, the following inequality holds for all coprime n and m:

$$\frac{|T_{(a_i)}(m/n) - T_{(a_i)}(a)|}{|m/n - a|} = \frac{T_{(a_i)}(m/n)}{|m/n - a|} \leq n^{\mu(a)+\varepsilon} a_n.$$

By choice of ε, this quantity tends to zero as n tends to infinity. It follows that $T'_{(a_i)}(a) = 0$.

Suppose $\mu(a) > \beta(a_i)$. Find $\varepsilon > 0$ such that $\mu(a) > \beta(a_i) + \varepsilon$. By definition,

$$\frac{|T_{(a_i)}(m/n) - T_{(a_i)}(a)|}{|m/n - a|} = \frac{T_{(a_i)}(m/n)}{|m/n - a|} > n^{\beta(a_i)+\varepsilon} a_n$$

has infinitely many solutions m and n. By the definition of $\beta(a_i)$, $\liminf_n n^{\beta(a_i)+\varepsilon} a_n > 0$. Reasoning as in the proof of Proposition 4.1, we see that $T_{(a_i)}$ is not differentiable at a. ■

We finish by remarking on some obvious consequences of the previous propositions. First, for $k \leq 2$, $T_{(1/n^k)}$ is nowhere differentiable. By Roth's Theorem, if $\alpha(a_n) > 2$, $T_{(a_i)}$ is differentiable on the set of algebraic irrational numbers. $T_{(1/n^9)}$ is differentiable at all the algebraic irrationals, $e, \pi, \pi^2, \ln(2)$, and $\zeta(3)$, and not differentiable on the set of Liouville numbers. Finally, if $\alpha(a_i) = \infty$, $T_{(a_i)}$ is differentiable on the set of all non-Liouville numbers. Since the set of Liouville numbers has measure zero, $T_{(a_i)}$ is differentiable almost everywhere.

5. CONCLUDING REMARKS. The motivating question of this note, as to whether there is a modification of Thomae's function that is differentiable on the irrationals, arose while the first author was teaching a one-semester course in real analysis. This question's connection to Diophantine approximation theory was not discovered until after the realization of Proposition 3.2. After the submission of the current manuscript, the authors were informed that a slightly less general version of Proposition 4.2 can be found in [**9**, p. 232].

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Over-iterates of Bernstein's Operators: A Short and Elementary Proof

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1. THE BERNSTEIN OPERATORS. Bernstein polynomials are an interesting topic that connects several areas of mathematics: approximation theory and probability theory, among others. They have many properties and applications, and the basic material is usually included in many undergraduate books.